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Positive solutions of m-point integral boundary value problems for second-order p-Laplacian dynamic equations on time scales

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Abstract

In this article, we use the Krasnosel'skii fixed point theorem, the Avery-Henderson fixed point theorem and the Leggett-Williams fixed point theorem to obtain some results for the existence of at least one, two or three positive solutions of m-point integral boundary value problems for nonlinear second-order p-Laplacian dynamic equations on time scales. Two examples are presented to illustrate the applications of the results.

MSC: 34B15; 34N05

Keywords: positive solution; p-Laplacian; time scales; fixed point theorem; integral boundary condition

1 Introduction

Analysis on measure chains was initiated by Stefan Hilger [1] as a bridge between continuous and discrete calculus. Dynamic equations on time scales have been a component of applied analysis on measure chains to describe the processes that feature both continuous and discrete elements [2–6]. This subject not only gives a unified approach to the study of differential and difference equations, but also gives an extended approach to the study of dynamic equations with nonuniform step size or a combination of real and discrete domains. Further, the study of time scale equations has led to several important applications, *e.g.*, in the study of economics, insect population models, heat transfer, stock market and epidemic models (see [7–10]), *etc.* Integral boundary value problems occur in the study of nonlocal phenomena in many different areas of applied mathematics, physics and engineering, *e.g.*, in heat conduction, chemical engineering, underground water flow, thermo-elasticity, plasma physics, *etc.* (see [11–15] and the references therein).

Throughout this paper, we denote the one-dimensional p-Laplacian operator by $\varphi_p(u)$, *i.e.*, $\varphi_p(u) = |u|^{p-2}u$ for $p > 1$ with $\varphi_p^{-1} = \varphi_q$, where $1/p + 1/q = 1$. For convenience, we make the blanket assumption that 0, T are points in a time scale $\mathbb{T}$; for an interval $(0, T)_{\mathbb{T}}$, we always mean $(0, T) \cap \mathbb{T}$. Other types of an interval are defined similarly.

In 2007, Sun and Li [16] discussed the existence of at least one, two or three positive solutions of the following boundary value problem:

$$\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\Delta} + h(t)f\left(u^{\sigma}(t)\right) = 0, \quad t \in [a, b]_{\mathbb{T}}, \tag{1.1}$$

$$u(a) - B_0\left(u^{\Delta}(a)\right) = 0, \quad u^{\Delta}\left(\sigma(b)\right) = 0. \tag{1.2}$$

They used the Krasnosel'skii fixed point theorem, the Avery-Henderson fixed point theorem and the Leggett-Williams fixed point theorem to prove the existence of multiple positive solutions to problem (1.1)-(1.2).

In 2009, Zhang and Qiao [17] studied the existence criteria for the m-point boundary value problem:

$$\bigl(\varphi_p\bigl(u^\triangle(t)\bigr)\bigr)^\triangle + a(t)f\bigl(t,u(t)\bigr) = 0, \quad t \in [0,1]_\mathbb{T}, \tag{1.3}$$

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i). \tag{1.4}$$

They obtained some results for the existence of multiple positive solutions of problem (1.3)-(1.4) by using the Krasnosel'skii fixed point theorem, the Avery-Henderson fixed point theorem and the Leggett-Williams fixed point theorem.

In 2011, Li and Zhang [18] considered the existence of at least three positive solutions for the boundary value problem with integral boundary conditions:

$$\bigl(\varphi_p\bigl(x^\triangle(t)\bigr)\bigr)^\nabla + \lambda f\bigl(t,x(t),x^\triangle(t)\bigr) = 0, \quad t \in (0,T)_\mathbb{T}, \tag{1.5}$$

$$x^\triangle(0) = 0, \qquad \alpha x(T) - \beta x(0) = \int_0^T g(s)x(s)\nabla s. \tag{1.6}$$

They established some sufficient conditions for the existence of positive solutions to problem (1.5)-(1.6) by using the Legget-Williams fixed point theorem. For some recent results on the existence of positive solutions for p-Laplacian dynamic equations on time scales, see [19–27]. However, to the best of the authors' knowledge, existence results for positive solutions of m-point integral boundary value problems for nonlinear p-Laplacian dynamic equations on time scales have not been studied.

In this article, we are concerned with the existence of multiple positive solutions to the m-point integral boundary value problem for a second-order p-Laplacian dynamic equation on time scale $\mathbb{T}$:

$$\bigl(\varphi_p\bigl(u^\triangle(t)\bigr)\bigr)^\triangle + a(t)f\bigl(t,u(t)\bigr) = 0, \quad t \in [0,1]_\mathbb{T}, \tag{1.7}$$

$$u^\triangle(0) = 0, \qquad u(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\xi_{i-1}}^{\xi_i} u(s)\Delta s, \tag{1.8}$$

where $\mathbb{T}$ is a time scale, $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \xi_{m-1} = 1$ and

(H_1) $0 < \sum_{i=1}^{m-1} \alpha_i(\xi_i - \xi_{i-1}) < 1$ such that $\alpha_i \geq 0$ for $i \in \{1,2,\dots,m-3\} \cup \{m-1\}$, $\alpha_{m-2} > 0$;
(H_2) $f \in C_{rd}([0,1]_\mathbb{T} \times [0,\infty),[0,\infty))$;
(H_3) $a \in C_{rd}([0,1]_\mathbb{T},[0,\infty))$ and there exists $t_0 \in (\xi_{m-2},1)_\mathbb{T}$ such that $a(t_0) > 0$.

The rest of the paper is organized as follows. In Section 2, we state and prove some lemmas which are used later. In Section 3, we use the Krasnosel'skii [28] fixed point theorem to obtain the existence of at least one positive solution of problem (1.7)-(1.8). In Section 4, by using the Avery-Henderson [29] fixed point theorem, we establish sufficient conditions for the existence of at least two positive solutions of problem (1.7)-(1.8). In Section 5, the existence of at least three positive solutions of problem (1.7)-(1.8) are proved by using the Leggett-Williams [30] fixed point theorem. Two illustrative examples are given in Section 6.

For convenience, we list the following well-known definitions which can be found in [4] and the references therein.

Definition 1.1 A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with topology and ordering inherited from $\mathbb{R}$.

The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} | s < t\}, \qquad \mu(t) := \sigma(t) - t,$$

for all $t \in \mathbb{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(t) < t$, t is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(t) = t$, t is said to be left dense. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^k = \mathbb{T}$.

Definition 1.2 A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous (rd-continuous is short for right-dense continuous) provided it is continuous at each right-dense point in $\mathbb{T}$ and has a left-sided limit at each left-dense point in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 1.3 For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the delta derivative of f at the point t is defined to be the number $f^\triangle(t)$ (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f\big(\sigma(t)\big) - f(s) - f^\triangle(t)\big(\sigma(t) - s\big)\right| \le \epsilon \left|\sigma(t) - s\right|$$

for all $s \in U$.

Definition 1.4 For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative is defined at the point t by

$$f^\triangle(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is not right-scattered, then the derivative is defined by

$$f^\triangle(t) = \lim_{s\to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

provided this limit exists.

Definition 1.5 If $F^\triangle(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

2 Preliminaries

In this section, we first prove and recall some lemmas which are used in what follows.

Lemma 2.1 *Let* $\sum_{i=1}^{m-1}\alpha_i(\xi_i-\xi_{i-1})\neq 1$. *Then, for* $y\in C_{rd}([0,1]_{\mathbb{T}},\mathbb{R})$, *the problem*

$$\big(\varphi_p\big(u^{\triangle}(t)\big)\big)^{\triangle}+y(t)=0,\quad t\in[0,1]_{\mathbb{T}}, \tag{2.1}$$

$$u^{\triangle}(0)=0,\qquad u(1)=\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}u(s)\Delta s, \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t)={}&-\int_0^t\varphi_q\left(\int_0^\tau y(s)\Delta s\right)\Delta\tau\\ &-\frac{1}{1-\sum_{i=1}^{m-1}\alpha_i(\xi_i-\xi_{i-1})}\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta\varphi_q\left(\int_0^\tau y(s)\Delta s\right)\Delta\tau\Delta\eta\\ &+\frac{1}{1-\sum_{i=1}^{m-1}\alpha_i(\xi_i-\xi_{i-1})}\int_0^1\varphi_q\left(\int_0^\tau y(s)\Delta s\right)\Delta\tau. \end{aligned} \tag{2.3}$$

Proof Integrating (2.1) from 0 to t and using the first condition of (2.2), one gets

$$u^{\triangle}(t)=-\varphi_q\left(\int_0^t y(s)\Delta s\right). \tag{2.4}$$

Integrating (2.4) from 0 to t, we obtain

$$u(t)=u(0)-\int_0^t\varphi_q\left(\int_0^\tau y(s)\Delta s\right)\Delta\tau. \tag{2.5}$$

In particular, for $t=1$, we have

$$u(1)=u(0)-\int_0^1\varphi_q\left(\int_0^\tau y(s)\Delta s\right)\Delta\tau.$$

Using the second condition of (2.2), we get that

$$\begin{aligned} &u(0)-\int_0^1\varphi_q\left(\int_0^\tau y(s)\Delta s\right)\Delta\tau\\ &\quad=u(0)\sum_{i=1}^{m-1}\alpha_i(\xi_i-\xi_{i-1})-\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta\varphi_q\left(\int_0^\tau y(s)\Delta s\right)\Delta\tau\Delta\eta. \end{aligned}$$

Hence,

$$\begin{aligned} u(0)={}&\frac{1}{1-\sum_{i=1}^{m-1}\alpha_i(\xi_i-\xi_{i-1})}\left[\int_0^1\varphi_q\left(\int_0^\tau y(s)\Delta s\right)\Delta\tau\right.\\ &\left.-\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta\varphi_q\left(\int_0^\tau y(s)\Delta s\right)\Delta\tau\Delta\eta\right]. \end{aligned}$$

Substituting the value of $u(0)$ in (2.5), we obtain the solution (2.3). □

Lemma 2.2 *Let* $\sum_{i=1}^{m-1}\alpha_i(\xi_i-\xi_{i-1})\neq 1$. *If* $y\in C_{rd}([0,1]_{\mathbb{T}},[0,\infty))$, *then the unique solution* u *of problem* (2.1)-(2.2) *satisfies*

$$u^{\Delta}(t)\leq 0,\qquad u^{\Delta\Delta}(t)\leq 0,\quad t\in[0,1]_{\mathbb{T}}.$$

Proof From (2.4), we have $u^{\Delta}(t)\leq 0$ for $t\in[0,1]_{\mathbb{T}}$. In fact, $\varphi_q(x)$ is a monotone increasing continuously differentiable function and

$$\left(\int_0^t y(s)\Delta s\right)^{\Delta}=y(t)\geq 0.$$

Then, by the chain rule [4], we get $u^{\Delta\Delta}(t)\leq 0$ for $t\in[0,1]_{\mathbb{T}}$. □

Lemma 2.3 *Let* $0<\sum_{i=1}^{m-1}\alpha_i(\xi_i-\xi_{i-1})<1$. *If* $y\in C_{rd}([0,1]_{\mathbb{T}},[0,\infty))$, *then the unique solution* u *of problem* (2.1)-(2.2) *satisfies*

$$u(t)\geq 0,\quad t\in[0,1]_{\mathbb{T}}.$$

Proof From Lemma 2.2, $u^{\Delta}(t)\leq 0$ for $t\in[0,1]_{\mathbb{T}}$, we know that u is nonincreasing on $[0,1]_{\mathbb{T}}$. Consequently, for each $t_1,t_2\in\mathbb{T}$ and $t_1\leq t_2$, it holds that $u(t_1)\geq u(t_2)$.

Therefore,

$$u(0)\geq u(\xi_1)\geq\cdots\geq u(\xi_{i-1})\geq u(\xi_i)\geq\cdots\geq u(\xi_{m-2})\geq u(1). \tag{2.6}$$

If $u(1)<0$, then the second condition of (2.2) together with (2.6) implies that

$$\begin{aligned}u(1)&=\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}u(s)\Delta s\geq\sum_{i=1}^{m-1}\alpha_i u(\xi_i)(\xi_i-\xi_{i-1})\\&\geq u(1)\sum_{i=1}^{m-1}\alpha_i(\xi_i-\xi_{i-1}).\end{aligned}$$

This contradicts the fact that $0<\sum_{i=1}^{m-1}\alpha_i(\xi_i-\xi_{i-1})<1$.

If $u(0)<0$, it follows that $u(1)<0$ since u is nonincreasing. Hence, we get a contradiction. Indeed, if $u(0)<0$ and $u(1)<0$, we again obtain a contradiction. □

Lemma 2.4 *Let* $\sum_{i=1}^{m-1}\alpha_i(\xi_i-\xi_{i-1})>1$. *If* $y\in C_{rd}([0,1]_{\mathbb{T}},[0,\infty))$, *then problem* (2.1)-(2.2) *has no positive solutions.*

Proof Suppose that problem (2.1)-(2.2) has a positive solution u satisfying $u(t)\geq 0$ for $t\in[0,1]_{\mathbb{T}}$. Then $u(\xi_i)\geq 0$ for all $i=1,\dots,m-1$. By the second condition of (2.2) and (2.6), we have

$$\begin{aligned}u(1)&=\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}u(s)\Delta s\\&\geq\sum_{i=1}^{m-1}\alpha_i u(\xi_i)(\xi_i-\xi_{i-1})\end{aligned}$$

$$\begin{aligned}
&\geq u(1)\sum_{i=1}^{m-1}\alpha_i(\xi_i-\xi_{i-1}) \\
&> u(1),
\end{aligned}$$

getting a contradiction. □

Let E denote the Banach space $C_{rd}[0,1]_{\mathbb{T}}$ with the norm $\|u\| = \sup_{t\in[0,1]_{\mathbb{T}}}|u(t)|$. Define the cone $P \subset E$, by

$$\begin{aligned}
P = \Bigg\{ & u \in E \mid u(t) \geq 0, u^{\Delta}(t) \leq 0, u^{\Delta\Delta}(t) \leq 0 \text{ for } t \in [0,1]_{\mathbb{T}}, \\
& \text{and } u^{\Delta}(0) = 0, u(1) = \sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} u(s)\Delta s \Bigg\}.
\end{aligned} \tag{2.7}$$

Define the operator $A : P \to E$ by

$$\begin{aligned}
Au(t) = &-\int_0^t \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&-\frac{1}{1-\Lambda}\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta \\
&+\frac{1}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau,
\end{aligned} \tag{2.8}$$

where a positive constant $\Lambda = \sum_{i=1}^{m-1}\alpha_i(\xi_i-\xi_{i-1}) < 1$. In view of Lemma 2.1, the solutions of problem (1.7)-(1.8) are given by the operator equation, $u(t) = Au(t)$.

From (2.8), we claim that for each $u \in P$, $Au \in P$ and satisfies (1.8). In fact, for $t \in [0,1]_{\mathbb{T}}$, we get

$$\begin{aligned}
Au(t) &\geq Au(1) \\
&= -\int_0^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\quad -\frac{1}{1-\Lambda}\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta \\
&\quad +\frac{1}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&= \frac{\Lambda}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\quad -\frac{1}{1-\Lambda}\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta \geq 0.
\end{aligned}$$

This implies that $Au(t) \geq 0$ for $t \in [0,1]_{\mathbb{T}}$. As in Lemma 2.2, we can prove that $(Au)^{\Delta}(t) \leq 0$, $(Au)^{\Delta\Delta}(t) \leq 0$ for $t \in [0,1]_{\mathbb{T}}$. In addition, we find that $(Au)^{\Delta}(0) = 0$ and $(Au)(1) = \sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i} Au(s)\Delta s$. So, $A : P \to P$. It is also easy to check that $A : P \to P$ is completely continuous.

Lemma 2.5 *Let* (H_1) *hold. If* $u \in P$, *then*

$$\min_{t\in[0,1]_{\mathbb{T}}} u(t) \geq \gamma \|u\|, \tag{2.9}$$

where

$$\gamma = \frac{\alpha_{m-2}(\xi_{m-2}-\xi_{m-3})(1-\xi_{m-2})}{1-\alpha_{m-2}\xi_{m-2}(\xi_{m-2}-\xi_{m-3})}, \tag{2.10}$$

which $\gamma > 0$.

Proof Since $u^{\Delta}(t) \leq 0$ for $t \in [0,1]_{\mathbb{T}}$, we have $\|u\| = u(0)$, $\min_{t\in[0,1]_{\mathbb{T}}} u(t) = u(1)$.
Thus,

$$u(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\xi_{i-1}}^{\xi_i} u(s)\Delta s \geq \sum_{i=1}^{m-1} \alpha_i u(\xi_i)(\xi_i - \xi_{i-1}) \geq \alpha_{m-2} u(\xi_{m-2})(\xi_{m-2}-\xi_{m-3}). \tag{2.11}$$

From $u^{\Delta\Delta}(t) \leq 0$ for $t \in [0,1]_{\mathbb{T}}$ and (2.11), we get

$$\begin{aligned} u(0) &\leq u(1) + \frac{u(1)-u(\xi_{m-2})}{1-\xi_{m-2}}(0-1) \\ &\leq u(1)\left[1 - \frac{1}{1-\xi_{m-2}} + \frac{1}{\alpha_{m-2}(\xi_{m-2}-\xi_{m-3})(1-\xi_{m-2})}\right] \\ &= u(1)\left[\frac{1-\alpha_{m-2}\xi_{m-2}(\xi_{m-2}-\xi_{m-3})}{\alpha_{m-2}(\xi_{m-2}-\xi_{m-3})(1-\xi_{m-2})}\right]. \end{aligned}$$

This implies that

$$\min_{t\in[0,1]_{\mathbb{T}}} u(t) \geq \frac{\alpha_{m-2}(\xi_{m-2}-\xi_{m-3})(1-\xi_{m-2})}{1-\alpha_{m-2}\xi_{m-2}(\xi_{m-2}-\xi_{m-3})} \|u\|.$$

Note that (H_1) yields

$$0 < 1 - \sum_{i=1}^{m-1} \alpha_i(\xi_i - \xi_{i-1}) < 1 - \alpha_{m-2}(\xi_{m-2}-\xi_{m-3}) < 1 - \alpha_{m-2}\xi_{m-2}(\xi_{m-2}-\xi_{m-3}).$$

Thus we have $\gamma > 0$. The proof of Lemma 2.5 is complete. □

In the following, for the sake of convenience, we set constants

$$L = \frac{1-\Lambda}{\int_0^1 \varphi_q(\int_0^\tau a(s)\Delta s)\Delta\tau}, \tag{2.12}$$

$$M = \frac{1-\Lambda}{\gamma \xi_{m-2} \int_{\xi_{m-2}}^1 \varphi_q(\int_{\xi_{m-2}}^\tau a(s)\Delta s)\Delta\tau}, \tag{2.13}$$

$$N = \frac{1-\Lambda}{\gamma \Lambda \int_{\xi_{m-2}}^1 \varphi_q(\int_{\xi_{m-2}}^\tau a(s)\Delta s)\Delta\tau}. \tag{2.14}$$

3 Existence of at least one positive solution

Now we are in a position to establish the main result. Our first result is based on the Krasnosel'skii fixed point theorem.

Theorem 3.1 (see [28]) *Let E be a Banach space, and let $P \subset E$ be a cone. Assume that Ω_1 and Ω_2 are bounded open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Au\| \le \|u\|$ *for* $u \in P \cap \partial\Omega_1$, $\|Au\| \ge \|u\|$ *for* $u \in P \cap \partial\Omega_2$; *or*

(ii) $\|Au\| \ge \|u\|$ *for* $u \in P \cap \partial\Omega_1$, $\|Au\| \le \|u\|$ *for* $u \in P \cap \partial\Omega_2$ *hold.*

Then A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 3.2 *Assume that* (H_1)-(H_3) *hold. In addition, suppose that there exist numbers $0 < r < R < \infty$ such that*

(A_1) $f(t,u) \le \varphi_p(L)\varphi_p(r)$ *for* $t \in [0,1]_{\mathbb{T}}$ *and* $0 \le u \le r$;

(A_2) $f(t,u) \ge \varphi_p(Mr)\varphi_p(R)$ *for* $t \in [\xi_{m-2},1]_{\mathbb{T}}$ *and* $R \le u < \infty$,

where constants L, M are defined by (2.12) *and* (2.13), *respectively.*

Then problem (1.7)-(1.8) *has at least one positive solution.*

Proof Firstly, we define a cone P and a completely continuous operator $A : P \to P$ as in (2.7) and (2.8), respectively.

Let $\Omega_1 = \{u \in C_{rd}([0,1]_{\mathbb{T}}) : \|u\| < r\}$. For any $u \in P \cap \partial\Omega_1$ with $\|u\| = r$, from condition (A_1), we obtain

$$\begin{aligned} Au(t) &= -\int_0^t \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\ &\quad - \frac{1}{1-\Lambda}\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta \\ &\quad + \frac{1}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\ &\le \frac{1}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\ &\le \frac{\varphi_q(\varphi_p(L)\varphi_p(r))}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)\Delta s\right)\Delta\tau \\ &= \frac{rL}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)\Delta s\right)\Delta\tau = r = \|u\|. \end{aligned}$$

This implies that $\|Au\| \le \|u\|$ for $u \in P \cap \partial\Omega_1$.

Set $\Omega_2 = \{u \in C_{rd}([0,1]_{\mathbb{T}}) : \|u\| < R\}$. Since $u \in P \cap \partial\Omega_2$, it follows that $\min_{t\in[0,1]_{\mathbb{T}}} u(t) \ge \gamma\|u\| = \gamma R$. Hence from condition (A_2), for any $u \in P \cap \partial\Omega_2$, we have

$$\begin{aligned} \|Au\| &\ge Au(\xi_{m-2}) \\ &= -\int_0^{\xi_{m-2}} \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\ &\quad - \frac{1}{1-\Lambda}\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&= \frac{\int_0^1 \varphi_q(\int_0^\tau a(s)f(s,u(s))\Delta s)\Delta\tau - \int_0^{\xi_{m-2}} \varphi_q(\int_0^\tau a(s)f(s,u(s))\Delta s)\Delta\tau}{1-\Lambda} \\
&\quad + \frac{1}{1-\Lambda}\Bigg[\Lambda\int_0^{\xi_{m-2}} \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\quad - \sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta\Bigg] \\
&\ge \frac{1}{1-\Lambda}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\quad - \frac{1}{1-\Lambda}\int_{\xi_{m-2}}^1\int_{\xi_{m-2}}^\eta \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta \\
&\ge \frac{1}{1-\Lambda}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\quad - \frac{1}{1-\Lambda}\int_{\xi_{m-2}}^1\int_{\xi_{m-2}}^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta \\
&= \frac{\xi_{m-2}}{1-\Lambda}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\ge \frac{\xi_{m-2}}{1-\Lambda}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_{\xi_{m-2}}^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\ge \varphi_q\big(\varphi_p(M\gamma)\varphi_p(R)\big)\frac{\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q(\int_{\xi_{m-2}}^\tau a(s)\Delta s)\Delta\tau}{1-\Lambda} \\
&= \frac{M\gamma R}{1-\Lambda}\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_{\xi_{m-2}}^\tau a(s)\Delta s\right)\Delta\tau = \|u\|.
\end{aligned}
$$

Therefore, $\|Au\| \ge \|u\|$ for $u \in P \cap \partial\Omega_2$.

Thus, from Theorem 3.1, it follows that A has a fixed point u in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $r \le \|u\| \le R$. Therefore, problem (1.7)-(1.8) has at least one positive solution. □

4 Existence of at least two positive solutions

In this section, we obtain the existence of at least two positive solutions of problem (1.7)-(1.8) by using the Avery-Henderson fixed point theorem which is as follows.

Theorem 4.1 (see [29]) *Let P be a cone in a real Banach space E. Set*

$$P(\Phi,\rho_3) = \big\{u \in P \mid \Phi(u) < \rho_3\big\}.$$

Let ν and Φ be increasing nonnegative continuous functionals on P, and let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some $\rho_3 > 0$ and $N > 0$,

$$\Phi(u) \le \theta(u) \le \nu(u) \quad \textit{and} \quad \|u\| \le N\Phi(u)$$

for all $u \in \overline{P(\Phi, \rho_3)}$. *Suppose there exist a completely continuous operator* $A : \overline{P(\Phi, \rho_3)} \to P$ *and* $0 < \rho_1 < \rho_2 < \rho_3$ *such that*

$$\theta(\lambda u) = \lambda\theta(u) \quad \textit{for } 0 \le \lambda \le 1 \textit{ and } u \in \partial P(\theta, \rho_2),$$

and

(i) $\Phi(Au) > \rho_3$ *for all* $u \in \partial P(\Phi, \rho_3)$;
(ii) $\theta(Au) < \rho_2$ *for all* $u \in \partial P(\theta, \rho_2)$;
(iii) $P(\nu, \rho_1) \neq \emptyset$ *and* $\nu(Au) > \rho_1$ *for all* $u \in \partial P(\nu, \rho_1)$.

Then A has at least two fixed points u_1 *and* u_2 *belonging to* $\overline{P(\Phi, \rho_3)}$ *satisfying*

$$\rho_1 < \nu(u_1) \quad \textit{with } \theta(u_1) < \rho_2, \quad \textit{and} \quad \rho_2 < \theta(u_2) \quad \textit{with } \Phi(u_2) < \rho_3.$$

Define a constant $l \in (0,1)_{\mathbb{T}}$ such that $0 < \xi_{m-2} < l < 1$. Let Φ, θ and ν be increasing, nonnegative and continuous functionals on P, defined by

$$\Phi(u) = u(\xi_{m-2}), \qquad \theta(u) = u(\xi_{m-2}), \qquad \nu(u) = u(l).$$

Obviously, $\Phi(u) = \theta(u) \le \nu(u)$ for each $u \in P$. Moreover, Lemma 2.5 implies $\Phi(u) = u(\xi_{m-2}) \ge \gamma\|u\|$ for each $u \in P$. It is easy to see that $\theta(0) = 0$ and $\theta(\lambda u) = \lambda\theta(u)$ for all $0 \le \lambda \le 1$ and $u \in \partial P(\theta, \rho_2)$.

We can now prove the following theorem.

Theorem 4.2 *Assume that* (H_1)-(H_3) *hold, and suppose that there exist positive numbers* $\rho_1 < \rho_2 < \rho_3$ *such that the function f satisfies the following conditions*:

(B_1) $f(t,u) > \varphi_p(N\gamma)\varphi_p(\rho_1)$ *for* $t \in [\xi_{m-2}, l]_{\mathbb{T}}$ *and* $u \in [\gamma\rho_1, \rho_1]$;
(B_2) $f(t,u) < \varphi_p(L)\varphi_p(\rho_2)$ *for* $t \in [\xi_{m-2}, 1]_{\mathbb{T}}$ *and* $u \in [0, \rho_2]$;
(B_3) $f(t,u) > \varphi_p(M\gamma)\varphi_p(\rho_3)$ *for* $t \in [\xi_{m-2}, l]_{\mathbb{T}}$ *and* $u \in [\rho_3, (1/\gamma)\rho_3]$,

where constants L, M, N are defined by (2.12), (2.13) *and* (2.14), *respectively.*

Then problem (1.7)-(1.8) *has at least two positive solutions* u_1 *and* u_2 *such that* $\rho_1 < u_1(l)$ *with* $u_1(\xi_{m-2}) < \rho_2$ *and* $\rho_2 < u_2(\xi_{m-2})$ *with* $u_2(\xi_{m-2}) < \rho_3$.

Proof We now wish to prove that all of the conditions of Theorem 4.1 are satisfied. For this purpose, we define the cone P as (2.7) and a completely continuous operator $A : P \to P$ by (2.8).

To check condition (i) of Theorem 4.1, we choose $u \in \partial P(\Phi, \rho_3)$, then $\Phi(u) = \rho_3$. This implies that $\rho_3 \le \|u\| \le (1/\gamma)\Phi(u) = (1/\gamma)\rho_3$. For $t \in [\xi_{m-2}, 1]_{\mathbb{T}}$, we have $\rho_3 \le u(t) \le (1/\gamma)\rho_3$. From condition (B_3), we get that $f(t,u) > \varphi_p(M\gamma)\varphi_p(\rho_3)$ for $t \in [\xi_{m-2}, l]_{\mathbb{T}}$. Since $Au \in P$, we obtain

$$\begin{aligned}
\Phi(Au) &= (Au)(\xi_{m-2}) \\
&= -\int_0^{\xi_{m-2}} \varphi_q\left(\int_0^{\tau} a(s)f\big(s, u(s)\big)\Delta s\right)\Delta\tau \\
&\quad - \frac{1}{1-\Lambda}\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i}\int_0^{\eta}\varphi_q\left(\int_0^{\tau} a(s)f\big(s, u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{1-\Lambda}\int_0^1\varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&=\frac{\int_0^1\varphi_q(\int_0^\tau a(s)f(s,u(s))\Delta s)\Delta\tau-\int_0^{\xi_{m-2}}\varphi_q(\int_0^\tau a(s)f(s,u(s))\Delta s)\Delta\tau}{1-\Lambda} \\
&\quad+\frac{1}{1-\Lambda}\left[\Lambda\int_0^{\xi_{m-2}}\varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\right. \\
&\quad\left.-\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta\varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta\right] \\
&\geq\frac{1}{1-\Lambda}\int_{\xi_{m-2}}^1\varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\quad-\frac{1}{1-\Lambda}\int_{\xi_{m-2}}^1\int_{\xi_{m-2}}^\eta\varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta \\
&\geq\frac{1}{1-\Lambda}\int_{\xi_{m-2}}^1\varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\quad-\frac{1}{1-\Lambda}\int_{\xi_{m-2}}^1\int_{\xi_{m-2}}^1\varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta \\
&=\frac{\xi_{m-2}}{1-\Lambda}\int_{\xi_{m-2}}^1\varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\geq\frac{\xi_{m-2}}{1-\Lambda}\int_{\xi_{m-2}}^1\varphi_q\left(\int_{\xi_{m-2}}^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&>\varphi_q\big(\varphi_p(M\gamma)\varphi_p(\rho_3)\big)\frac{\xi_{m-2}\int_{\xi_{m-2}}^1\varphi_q(\int_{\xi_{m-2}}^\tau a(s)\Delta s)\Delta\tau}{1-\Lambda} \\
&=\frac{M\gamma\rho_3}{1-\Lambda}\xi_{m-2}\int_{\xi_{m-2}}^1\varphi_q\left(\int_{\xi_{m-2}}^\tau a(s)\Delta s\right)\Delta\tau \\
&=\rho_3.
\end{aligned}
$$

Hence, condition (i) of Theorem 4.1 holds.

We now prove that condition (ii) in Theorem 4.1 holds. In fact, for $u\in\partial P(\theta,\rho_2)$, we have $\theta(u)=\rho_2$. This implies that $0\leq u(t)\leq\|u\|\leq(1/\gamma)\rho_2$ for $t\in[\xi_{m-2},1]_{\mathbb{T}}$. From condition (B_2), we have

$$
\begin{aligned}
\theta(Au)&=(Au)(\xi_{m-2}) \\
&\leq\frac{1}{1-\Lambda}\int_0^1\varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&<\frac{\varphi_q(\varphi_p(L)\varphi_p(\rho_2))}{1-\Lambda}\int_0^1\varphi_q\left(\int_0^\tau a(s)\Delta s\right)\Delta\tau \\
&=\frac{L\rho_2}{1-\Lambda}\int_0^1\varphi_q\left(\int_0^\tau a(s)\Delta s\right)\Delta\tau \\
&=\rho_2=\|u\|.
\end{aligned}
$$

This shows that condition (ii) of Theorem 4.1 is satisfied.

Now, we assert that condition (iii) of Theorem 4.1 also holds. If $u(t) = \rho_1/2$ for $t \in [0,1]_{\mathbb{T}}$, then $\nu(u) = \rho_1/2$. Thus $P(\nu,\rho_1) \neq \emptyset$. Let $u \in \partial P(\nu,\rho_1)$, then $\nu(u) = u(l) = \rho_1$. So that $\gamma\rho_1 \leq u(t) \leq \|u\| \leq \rho_1$. From condition (B_1), for any $Au \in P$, we have

$$\begin{aligned}
\nu(Au) &= (Au)(l) \geq (Au)(1) \\
&= -\int_0^1 \varphi_q\left(\int_0^\tau a(s)f\bigl(s,u(s)\bigr)\Delta s\right)\Delta\tau \\
&\quad - \frac{1}{1-\Lambda}\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta \varphi_q\left(\int_0^\tau a(s)f\bigl(s,u(s)\bigr)\Delta s\right)\Delta\tau\,\Delta\eta \\
&\quad + \frac{1}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)f\bigl(s,u(s)\bigr)\Delta s\right)\Delta\tau \\
&= \frac{1}{1-\Lambda}\left[\Lambda\int_0^1 \varphi_q\left(\int_0^\tau a(s)f\bigl(s,u(s)\bigr)\Delta s\right)\Delta\tau\right. \\
&\quad \left. -\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta \varphi_q\left(\int_0^\tau a(s)f\bigl(s,u(s)\bigr)\Delta s\right)\Delta\tau\,\Delta\eta\right] \\
&= \frac{1}{1-\Lambda}\left[\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^1 \varphi_q\left(\int_0^\tau a(s)f\bigl(s,u(s)\bigr)\Delta s\right)\Delta\tau\,\Delta\eta\right. \\
&\quad \left. -\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta \varphi_q\left(\int_0^\tau a(s)f\bigl(s,u(s)\bigr)\Delta s\right)\Delta\tau\,\Delta\eta\right] \\
&= \frac{1}{1-\Lambda}\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_\eta^1 \varphi_q\left(\int_0^\tau a(s)f\bigl(s,u(s)\bigr)\Delta s\right)\Delta\tau\,\Delta\eta \\
&\geq \frac{1}{1-\Lambda}\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_0^\tau a(s)f\bigl(s,u(s)\bigr)\Delta s\right)\Delta\tau\,\Delta\eta \\
&\geq \frac{\Lambda}{1-\Lambda}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_{\xi_{m-2}}^\tau a(s)f\bigl(s,u(s)\bigr)\Delta s\right)\Delta\tau \\
&> \varphi_q\bigl(\varphi_p(N\gamma)\varphi_p(\rho_1)\bigr)\frac{\Lambda\int_{\xi_{m-2}}^1 \varphi_q(\int_{\xi_{m-2}}^\tau a(s)\Delta s)\Delta\tau}{1-\Lambda} \\
&= \frac{N\gamma\rho_1\Lambda}{1-\Lambda}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_{\xi_{m-2}}^\tau a(s)\Delta s\right)\Delta\tau = \rho_1.
\end{aligned}$$

Therefore, condition (iii) of Theorem 4.1 is satisfied.

Thus, by Theorem 4.1, problem (1.7)-(1.8) has at least two positive solutions u_1 and u_2 such that $\rho_1 < u_1(l)$ with $u_1(\xi_{m-2}) < \rho_2$ and $\rho_2 < u_2(\xi_{m-2})$ with $u_2(\xi_{m-2}) < \rho_3$. □

5 Existence of at least three positive solutions

In this section, we use the Leggett-Williams fixed point theorem to prove the existence of at least three positive solutions to problem (1.7)-(1.8). The Leggett-Williams fixed point theorem is as follows.

Theorem 5.1 (see [30]) *Let P be a cone in the real Banach space E. Set*

$$P_r = \{x \in P \mid \|x\| < r\}, \qquad P(\Psi, a, b) = \{x \in P \mid a \le \Psi(x), \|x\| \le b\}.$$

Let $A : \overline{P}_r \to \overline{P}_r$ be a completely continuous operator and let Ψ be a nonnegative continuous concave functional on P with $\Psi(u) \le \|u\|$ for all $u \in \overline{P}_r$. Suppose that there exists $0 < \rho_1 < \rho_2 < (1/\gamma)\rho_2 < \rho_3$ such that the following conditions hold:

(i) $\{u \in P(\Psi, \rho_2, (1/\gamma)\rho_2) \mid \Psi(u) > \rho_2\} \neq \emptyset$ *and* $\Psi(Au) > \rho_2$ *for all* $u \in \partial P(\Psi, \rho_2, (1/\gamma)\rho_2)$;

(ii) $\|Au\| < \rho_1$ *for* $\|u\| \le \rho_1$;

(iii) $\Psi(Au) > \rho_2$ *for* $u \in P(\Psi, \rho_2, \rho_3)$ *with* $\|Au\| > (1/\gamma)\rho_2$.

Then A has at least three fixed points u_1, u_2 and u_3 in $\overline{P}_r$ satisfying $\|u_1\| < \rho_1$, $\Psi(u_2) > \rho_2$, $\rho_1 < \|u_3\|$ with $\Psi(u_3) < \rho_2$.

We now prove the following result.

Theorem 5.2 *Assume that* (H_1)-(H_3) *hold. Suppose that there exist constants $0 < \rho_1 < \rho_2 < (1/\gamma)\rho_2 \le \rho_3$ such that*

(C_1) $f(t,u) \le \varphi_p(L)\varphi_p(\rho_3)$ *for* $t \in [\xi_{m-2}, 1]_{\mathbb{T}}$ *and* $u \in [0, \rho_3]$;
(C_2) $f(t,u) > \varphi_p(M\gamma)\varphi_p(\rho_2)$ *for* $t \in [\xi_{m-2}, 1]_{\mathbb{T}}$ *and* $u \in [\rho_2, (1/\gamma)\rho_2]$;
(C_3) $f(t,u) < \varphi_p(L)\varphi_p(\rho_1)$ *for* $t \in [\xi_{m-2}, 1]_{\mathbb{T}}$ *and* $u \in [0, \rho_1]$,

where constants L, M are defined by (2.12) *and* (2.13), *respectively.*

Then problem (1.7)-(1.8) *has at least three positive solutions u_1, u_2 and u_3 such that $\|u_1\| < \rho_1$, $u_2(\xi_{m-2}) > \rho_2$, $\|u_3\| > \rho_1$ with $u_3(\xi_{m-2}) < \rho_2$.*

Proof We will show that all the conditions of Leggett-Williams Theorem 5.1 hold with respect to the operator A defined in (2.8).

At first, we define a nonnegative continuous concave functional $\Psi : P \to [0,\infty)$ by $\Psi(u) = u(\xi_{m-2})$, where the cone P is defined by (2.7). In fact, for $u \in P$, we get $\Psi(u) \le \|u\|$. If $u \in \overline{P}_{\rho_3}$, then $\|u\| \le \rho_3$. From condition (C_1), we obtain

$$\begin{aligned}
Au(t) &= -\int_0^t \varphi_q\left(\int_0^\tau a(s)f(s,u(s))\Delta s\right)\Delta\tau \\
&\quad - \frac{1}{1-\Lambda}\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i}\int_0^\eta \varphi_q\left(\int_0^\tau a(s)f(s,u(s))\Delta s\right)\Delta\tau\,\Delta\eta \\
&\quad + \frac{1}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)f(s,u(s))\Delta s\right)\Delta\tau \\
&\le \frac{1}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)f(s,u(s))\Delta s\right)\Delta\tau \\
&\le \frac{\varphi_q(\varphi_p(L)\varphi_p(\rho_3))}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)\Delta s\right)\Delta\tau \\
&= \frac{L\rho_3}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)\Delta s\right)\Delta\tau = \rho_3.
\end{aligned}$$

This implies that $\|Au\| \le \rho_3$. Therefore, we have $A : \overline{P}_{\rho_3} \to \overline{P}_{\rho_3}$. Since $(\rho_2/\gamma) \in P(\Psi, \rho_2, (\rho_2/\gamma))$ and $\Psi((\rho_2/\gamma)) = (\rho_2/\gamma) > \rho_2$, then $\{u \in P(\Psi, \rho_2, (\rho_2/\gamma)) \mid \Psi(u) > \rho_2\} \neq \emptyset$.

For $u \in P(\Psi, \rho_2, (\rho_2/\gamma))$, we get $\rho_2 \le u(\xi_{m-2}) \le \|u\| \le (\rho_2/\gamma)$. By using condition ($C_2$), we obtain

$$\begin{aligned}
\Psi(Au) &= (Au)(\xi_{m-2}) \\
&= -\int_0^{\xi_{m-2}} \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\quad - \frac{1}{1-\Lambda}\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta \\
&\quad + \frac{1}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&= \frac{\int_0^1 \varphi_q(\int_0^\tau a(s)f(s,u(s))\Delta s)\Delta\tau - \int_0^{\xi_{m-2}} \varphi_q(\int_0^\tau a(s)f(s,u(s))\Delta s)\Delta\tau}{1-\Lambda} \\
&\quad + \frac{1}{1-\Lambda}\left[\Lambda\int_0^{\xi_{m-2}} \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\right. \\
&\quad \left. - \sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}\int_0^\eta \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta\right] \\
&\ge \frac{1}{1-\Lambda}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\quad - \frac{1}{1-\Lambda}\int_{\xi_{m-2}}^1\int_{\xi_{m-2}}^\eta \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta \\
&\ge \frac{1}{1-\Lambda}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\quad - \frac{1}{1-\Lambda}\int_{\xi_{m-2}}^1\int_{\xi_{m-2}}^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau\,\Delta\eta \\
&= \frac{\xi_{m-2}}{1-\Lambda}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_0^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&\ge \frac{\xi_{m-2}}{1-\Lambda}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_{\xi_{m-2}}^\tau a(s)f\big(s,u(s)\big)\Delta s\right)\Delta\tau \\
&> \varphi_q\big(\varphi_p(M\gamma)\varphi_p(\rho_2)\big)\frac{\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q(\int_{\xi_{m-2}}^\tau a(s)\Delta s)\Delta\tau}{1-\Lambda} \\
&= \frac{M\gamma\rho_2}{1-\Lambda}\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_{\xi_{m-2}}^\tau a(s)\Delta s\right)\Delta\tau = \rho_2.
\end{aligned}$$

Hence, condition (i) of Theorem 5.1 is satisfied.

Indeed, if $\|u\| \le \rho_1$, then condition (C_3) implies that

$$\begin{aligned}
(Au)(t) &< \frac{\varphi_q(\varphi_p(L)\varphi_p(\rho_1))}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)\Delta s\right)\Delta\tau \\
&= \frac{L\rho_1}{1-\Lambda}\int_0^1 \varphi_q\left(\int_0^\tau a(s)\Delta s\right)\Delta\tau = \rho_1.
\end{aligned}$$

Thus $\|Au\| < \rho_1$. Therefore, condition (ii) of Theorem 5.1 holds.

We finally show that condition (iii) of Theorem 5.1 also holds. Assume that $u \in P(\Psi, \rho_2, \rho_3)$, with $\|Au\| > (1/\gamma)\rho_2$. Then we obtain

$$\begin{aligned}\Psi(Au) &= (Au)(\xi_{m-2}) \\ &\geq (Au)(1) \\ &\geq \gamma\|Au\| > \rho_2.\end{aligned}$$

So, condition (iii) of Theorem 5.1 is satisfied. Therefore, an application of Theorem 5.1 implies that problem (1.7)-(1.8) has at least three positive solutions u_1, u_2 and u_3 such that $\|u_1\| < \rho_1$, $u_2(\xi_{m-2}) > \rho_2$ and $\|u_3\| > \rho_1$ with $u_3(\xi_{m-2}) < \rho_2$. □

6 Numerical examples

In this section, we present some examples to illustrate our results.

Example 6.1 Consider the following six-point integral boundary value problem with $p = 3$ and $\mathbb{T} = \mathbb{R}$:

$$\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\Delta} + f\left(t, u(t)\right) = 0, \quad t \in [0,1]_{\mathbb{T}}, \tag{6.1}$$

$$u^{\Delta}(0) = 0, \qquad u(1) = \frac{1}{4}\int_0^{1/5} u(s)\Delta s + \frac{1}{5}\int_{2/5}^{3/5} u(s)\Delta s + 2\int_{3/5}^{4/5} u(s)\Delta s, \tag{6.2}$$

where

$$f(t,u) = \begin{cases} \frac{1}{100}t + u^3, & t \in [0,1], u \in [0, \frac{1}{5}], \\ \frac{1}{100}t + u^3 + 100(u - \frac{1}{5})^{1/4}, & t \in [0,1], u \in [\frac{1}{5}, \frac{3}{5}], \\ \frac{1}{100}t + u^3 + 100(u - \frac{1}{5})^{1/4} + 10(u - \frac{3}{5}), & t \in [0,1], u \in [\frac{3}{5}, \infty). \end{cases}$$

Set $\alpha_1 = 1/4$, $\alpha_3 = 1/5$, $\alpha_4 = 2$, $\alpha_2 = \alpha_5 = 0$, $\xi_0 = 0$, $\xi_1 = 1/5$, $\xi_2 = 2/5$, $\xi_3 = 3/5$, $\xi_4 = 4/5$, $\xi_5 = 1$ and $a(t) = 1$. We can show that

$$\Lambda = \sum_{i=1}^{5} \alpha_i(\xi_i - \xi_{i-1}) = \frac{49}{100} < 1.$$

Through a simple calculation we can get

$$\begin{aligned}\gamma &= \frac{\alpha_{m-2}(\xi_{m-2} - \xi_{m-3})(1 - \xi_{m-2})}{1 - \alpha_{m-2}\xi_{m-2}(\xi_{m-2} - \xi_{m-3})} = \frac{2}{17}, \\ M &= \frac{1 - \Lambda}{\gamma\xi_{m-2}\int_{\xi_{m-2}}^{1} \varphi_q(\int_{\xi_{m-2}}^{\tau} a(s)\Delta s)\Delta\tau} = \frac{2601}{64}\sqrt{5}, \\ L &= \frac{1 - \Lambda}{\int_0^1 \varphi_q(\int_0^{\tau} a(s)\Delta s)\Delta\tau} = \frac{153}{200}.\end{aligned}$$

Choose $r = 1/5$ and $R = 3/5$, then $f(t,u)$ satisfies

$$f(t,u) \leq \frac{1}{100} + \left(\frac{1}{5}\right)^3 < \left(\frac{153}{200} \times \frac{1}{5}\right)^2 = \varphi_3(Lr), \quad t \in [0,1], u \in \left[0, \frac{1}{5}\right],$$

and

$$\begin{aligned} f(t,u) &\geq \frac{1}{100}\left(\frac{4}{5}\right) + \left(\frac{3}{5}\right)^3 + 100\left(\frac{3}{5} - \frac{1}{5}\right)^{1/4} \\ &> \left(\frac{2601\sqrt{5}}{64} \times \frac{2}{17} \times \frac{3}{5}\right)^2 = \varphi_3(M\gamma R), \quad t \in \left[\frac{4}{5}, 1\right], u \in \left[\frac{3}{5}, \infty\right). \end{aligned}$$

By Theorem 3.2, we have that boundary value problem (6.1)-(6.2) has at least one positive solution.

Example 6.2 Consider the following six-point integral boundary value problem with $p = 2$ and $\mathbb{T} = \{0\} \cup \{1/2^n : n \in \mathbb{N}\} \cup (\frac{1}{2}, 1]$ ($\mathbb{N}$ stands for the natural number set).

$$\bigl(\varphi_p\bigl(u^\Delta(t)\bigr)\bigr)^\Delta + f\bigl(t, u(t)\bigr) = 0, \quad t \in [0,1]_{\mathbb{T}}, \tag{6.3}$$

$$u^\Delta(0) = 0, \qquad u(1) = \frac{1}{4}\int_0^{1/16} u(s)\Delta s + \frac{1}{6}\int_{1/8}^{1/4} u(s)\Delta s + 3\int_{1/4}^{1/2} u(s)\Delta s, \tag{6.4}$$

where

$$f(t,u) = \begin{cases} \frac{1}{50}t + \frac{1}{100}u, \\ \quad t \in [\frac{1}{2}, 1], u \in [0,1], \\ \frac{1}{50}t + \frac{1}{100}u + 7(u-1)^{1/6}, \\ \quad t \in [\frac{1}{2}, 1], u \in [1,2], \\ \frac{1}{50}t + \frac{1}{100}u + 7(u-1)^{1/6} + \frac{1}{20}(u-2)^{1/2}, \\ \quad t \in [\frac{1}{2}, 1], u \in [2, \frac{4096}{585}], \\ \frac{1}{50}t + \frac{1}{100}u + 7(u-1)^{1/6} + \frac{1}{20}(u-2)^{1/2} + \frac{1}{40}(u - \frac{4096}{585}), \\ \quad t \in [\frac{1}{2}, 1], u \in [\frac{4096}{585}, 30]. \end{cases}$$

Set $\alpha_1 = 1/4$, $\alpha_3 = 1/6$, $\alpha_4 = 3$, $\alpha_2 = \alpha_5 = 0$, $\xi_0 = 0$, $\xi_1 = 1/16$, $\xi_2 = 1/8$, $\xi_3 = 1/4$, $\xi_4 = 1/2$, $\xi_5 = 1$ and $a(t) = 1$. We can show that

$$\Lambda = \sum_{i=1}^{5} \alpha_i(\xi_i - \xi_{i-1}) = \frac{151}{192} < 1.$$

Through a simple calculation we can get

$$\begin{aligned} \gamma &= \frac{\alpha_{m-2}(\xi_{m-2} - \xi_{m-3})(1 - \xi_{m-2})}{1 - \alpha_{m-2}\xi_{m-2}(\xi_{m-2} - \xi_{m-3})} = \frac{3}{5}, \\ M &= \frac{1 - \Lambda}{\gamma \xi_{m-2} \int_{\xi_{m-2}}^1 \varphi_q(\int_{\xi_{m-2}}^\tau a(s)\Delta s)\Delta\tau} = \frac{205}{36}, \\ L &= \frac{1 - \Lambda}{\int_0^1 \varphi_q(\int_0^\tau a(s)\Delta s)\Delta\tau} = \frac{41}{88}. \end{aligned}$$

Choose $\rho_1 = 1$, $\rho_2 = 2$ and $\rho_3 = 30$, then $f(t,u)$ satisfies

$$f(t,u) \leq \frac{1}{50} + \frac{1}{100} < \frac{41}{88} \times 1 = \varphi_2(L\rho_1), \quad t \in \left[\frac{1}{2}, 1\right], u \in [0,1],$$

and

$$\begin{aligned} f(t,u) &\geq \frac{1}{50}\left(\frac{1}{2}\right) + \frac{1}{100}(2) + 7(2-1)^{1/6} \\ &> \frac{205}{36} \times \frac{3}{5} \times 2 = \varphi_2(M\gamma\rho_2), \quad t \in \left[\frac{1}{2}, 1\right], u \in \left[2, \frac{4096}{585}\right], \end{aligned}$$

and

$$\begin{aligned} f(t,u) &\leq \frac{1}{50} + \frac{1}{100}(30) + 7(30-1)^{1/6} + \frac{1}{20}(30-2)^{1/2} + \frac{1}{40}\left(30 - \frac{4096}{585}\right) \\ &< \frac{41}{88} \times 30 = \varphi_2(L\rho_3), \quad t \in \left[\frac{1}{2}, 1\right], u \in [0, 30]. \end{aligned}$$

By Theorem 5.2, we get that problem (6.3)-(6.4) has at least three positive solutions u_1, u_2 and u_3 such that $\|u_1\| < 1$, $u_2(\frac{1}{2}) > 2$ and $\|u_3\| > 1$ with $u_3(\frac{1}{2}) < 2$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally in this article. They read and approved the final manuscript.

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